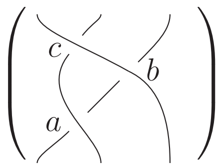
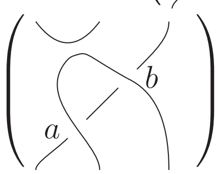
SOME CHAIN MAPS ON KHOVANOV COMPLEXES AND REIDEMEISTER MOVES

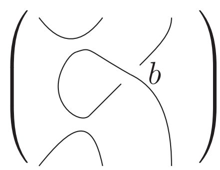
NOBORU ITO

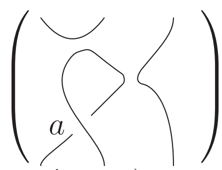
Abstract. We introduce some chain maps between Khovanov complexes. Each of the chain maps commutes with a chain homotopy map and a retraction maps which obtain a Reidemeister invariance of Khovanov homology.

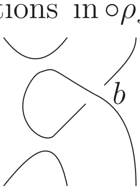
1. Introduction.

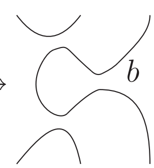
Let $\mathcal{C}_3 = \mathcal{C}$ 

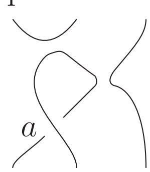
be a Khovanov complex of a link diagram and $\mathcal{C}_2 = \mathcal{C}$ 

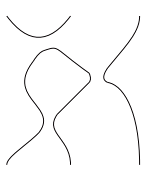
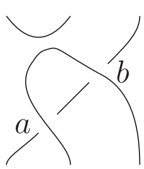
, $\mathcal{C}_1 = \mathcal{C}$ 

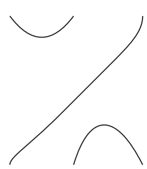
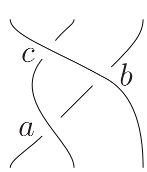
, and $\mathcal{C}_{1'} = \mathcal{C}$ 

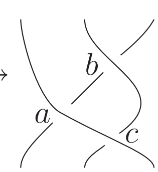
be its subcomplexes. There exist chain homotopy maps h_J ($J = 1,\ 1',\ 2,\ 3$) relating the identity maps $\mathcal{C}_J \to \mathcal{C}_J$ with compositions $\mathrm{in} \circ \rho_J$ of an inclusion maps in and a retraction ρ_J for Reidemeister moves 1: 

$\to$ 

, $1'$: 

$\to$ 
, 2: 

$\to$ 
, and 3: 

$\to$ 

(Section 4, Appendix A, B). This paper will be show that the natural chain maps π_J: $\mathcal{C}_J \to \mathcal{C}_{J-1}$ ($J = 2,\ 3$) satisfy the relations $h_{J-1} \circ \pi_J = \pi_J \circ h_J$ (Theorem 1) and similar relations for ρ_J (Theorem 2).

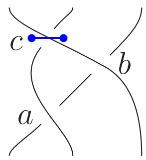
In section 1 chain maps π_J are defined. In section 2 relations of h_J, ρ_J, and π_J are given. In section 3 a map $\tilde{\pi}_2$ similar to π_2 is introduced. In section 4 we obtain the proof of the right twisted first Reidemeister invariance of Khovanov homology for a general differential. Appendix contains the definitions h_J and ρ_J provided by [1, 2]. All notations in this paper and the definition of the differential $\delta_{s,t}$ follows [2].

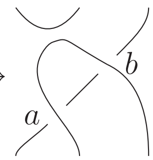
The author is a Research Fellow of the Japan Society for the Promotion of Science. This work was partly supported by KAKENHI.

2. The chain map π_J.

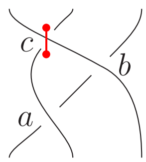
$\pi_3 : \mathcal{C}_3 \longrightarrow \mathcal{C}_2$ is defined by

(1)



$\mapsto$


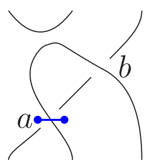
,

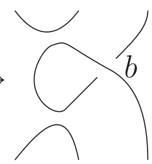


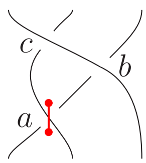
$\mapsto 0.$

$\pi_2 : \mathcal{C}_2 \longrightarrow \mathcal{C}_1$ is defined by

(2)



$\mapsto$




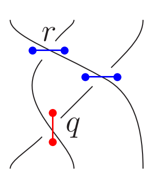
$\mapsto 0.$

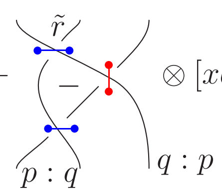
Theorem 1. *The chain maps π_3 and π_2 satisfy the following.*

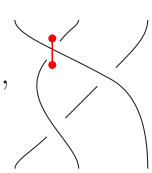
$$h_2 \circ \pi_3 = \pi_3 \circ h_3, \tag{3}$$

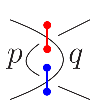
$$h_1 \circ \pi_2 = \pi_2 \circ h_2. \tag{4}$$

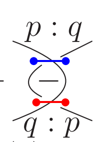
Proof. $\delta_{s,t} \circ \pi_J = \pi_J \circ \delta_{s,t}$ and $h_{J-1} \circ \pi_J = \pi_J \circ h_J$ $(J = 2,\, 3)$ are proved by direct computation for every generator of $\mathcal{C}_J$. $\square$

Let $\mathcal{C}'_3 = \mathcal{C}\Bigg($


$\otimes [xa] +$


$\otimes [xb],$

$\otimes [x]\Bigg)$ and

$\mathcal{C}'_2 = \mathcal{C}\Bigg($


$\otimes [xa] +$


$\otimes [xb]\Bigg)$. $\mathcal{C}'_J$ is a subcomplex of $\mathcal{C}_J$ $(J = 2,\, 3)$.

We define $\pi'_3 : \mathcal{C}'_3 \longrightarrow \mathcal{C}_2$ by (1).

Theorem 2.

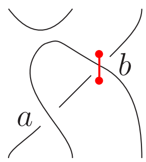
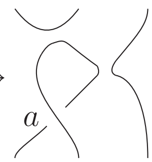
$$\rho_2 \circ \pi_3 = \pi'_3 \circ \rho_3. \tag{5}$$

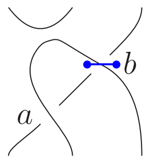
Proof. $\delta_{s,t} \circ \pi_3' = \pi_3' \circ \delta_{s,t}$ and $\rho_2 \circ \pi_3 = \pi_3' \circ \rho_3$ are proved by direct computation for every generator of $\mathcal{C}_3$. □

3. A similar map $\tilde{\pi}_2$ to π_2.

In this section we introduce a map $\tilde{\pi}_2$. It is not chain maps, but it has similar property $h_{1'} \circ \tilde{\pi}_2 = \tilde{\pi}_2 \circ h_{1'}$ (Theorem 3).

The map $\tilde{\pi}_2 : \mathcal{C}_2 \to \mathcal{C}_{1'}$ is defined by

(6)  $\mapsto$ 

(7)  $\mapsto 0.$

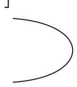
Theorem 3.

$$h_{1'} \circ \tilde{\pi}_2 = \tilde{\pi}_2 \circ h_{1'}. \tag{8}$$

Proof. $h_{1'} \circ \tilde{\pi}_2 = \tilde{\pi}_2 \circ h_{1'}$ is proved by direct computation for every generator of $\mathcal{C}_2$. □

4. Right twisted first Reidemeister invariance for the general differential $\delta_{s,t}$.

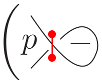
In this section we will show that the right twisted first Reidemeister invariance of Khovanov homology because the proof of this case is missing in [2].

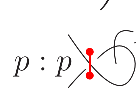
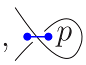
The right twisted first Reidemeister move is $D' =$  $\stackrel{1}{\sim}$  $= D$, we consider the composition

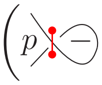
$$\mathcal{C}(D') = \mathcal{C} \oplus \mathcal{C}_{\text{contr}} \xrightarrow{\rho_1} \mathcal{C} \xrightarrow{\text{isom}} \mathcal{C}(D) \tag{9}$$

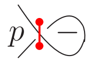
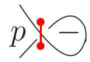
where a is a crossing and $\mathcal{C}$, $\mathcal{C}_{\text{contr}}$, ρ_1 and the isomorphism are defined in the following formulas (10)–(13).

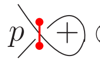
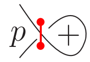
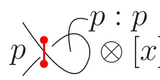
First,

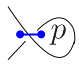
(10)
$\mathcal{C} := \mathcal{C}\Big($  $\otimes [x]\Big)$,

$\mathcal{C}_{\text{contr}} := \mathcal{C}\Big($  $\otimes [x]$,  $\otimes [xa]\Big)$.

Second, the retraction $\rho_1 : \mathcal{C}\Big($  $\Big) \to \mathcal{C}\Big($  $\otimes [x]\Big)$ is defined by the formulas

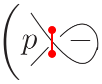
 $\otimes [x] \mapsto$  $\otimes [x],$

(11)  $\otimes [x] \mapsto$  $\otimes [x] -\ p : p$  $\otimes [x],$

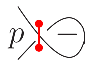
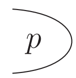
 $\otimes [xa] \mapsto\ 0.$

We can verify that $\delta_{s,t} \circ \rho_1 = \rho_1 \circ \delta_{s,t}$. Then ρ_1 is a chain map.

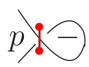
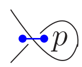
Third, the isomorphism

(12) $\mathcal{C}\Big($  $\otimes [x]\Big) \to \mathcal{C}\Big($  $\otimes [x]\Big)$

is defined by the formulas

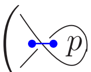
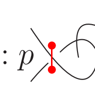
(13)  $\otimes [x] \mapsto$  $\otimes [x].$

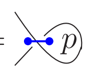
The homotopy connecting in $\circ\, \rho_1$ to the identity : $\mathcal{C}\Big($  $\Big) \to \mathcal{C}\Big($  $\Big)$ such that $\delta_{s,t} \circ h_1 + h_1 \circ \delta_{s,t} = \mathrm{id} - \mathrm{in} \circ \rho_1$, is defined by the formulas:

(14)  $\otimes [xa] \mapsto$  $\otimes [x]$, otherwise $\mapsto 0$.

Remark 1. *The explicit formula (14) of the homotopy map h_1 in the case ($s = t = 0$) of the original Khovanov homology is given by Oleg Viro* [3, Subsection 5.5].

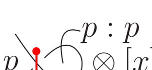
We can verify $\delta_{s,t} \circ h_1 + h_1 \circ \delta_{s,t} = \mathrm{id} - \mathrm{in} \circ \rho_1$ by a direct computation as follows.

(15) $(h_1 \circ \delta_{s,t} + \delta_{s,t} \circ h_1)\Big($  $\otimes [x]\Big) = h_1\Big(p : p$  $\otimes [xa]\Big)$

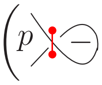
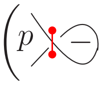
$=$  $\otimes [x]$

$= (\mathrm{id} - \rho_1)\Big($  $\otimes [x]\Big).$

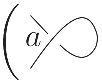
Similarly,

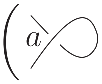
(16) $(h_1 \circ \delta_{s,t} + \delta_{s,t} \circ h_1)\Big($  $\otimes [x]\Big) =\ p : p$  $\otimes [x]$

$= (\mathrm{id} - \rho_1)\Big($  $\otimes [x]\Big),$

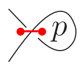
(17) $(h_1 \circ \delta_{s,t} + \delta_{s,t} \circ h_1)\Big($  $\otimes [x]\Big) = 0 = (\mathrm{id} - \rho_1)\Big($  $\otimes [x]\Big).$

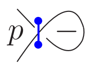
Appendix A. Chain homotopy maps.

The homotopy connecting $\mathrm{in} \circ \rho_{1'}$ to the identity $h_{1'} : \mathcal{C}\Big($ 

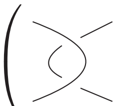
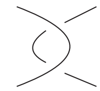
$\Big) \to \mathcal{C}\Big($ 

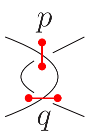
$\Big)$ such that $\delta_{s,t} \circ h_{1'} + h_{1'} \circ \delta_{s,t} = \mathrm{id} - \mathrm{in} \circ \rho_{1'}$, is defined by the formulas:

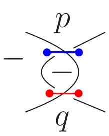
(18) 

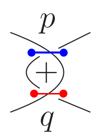
$\otimes [xa] \mapsto p$ 

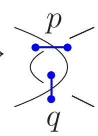
$\otimes [x]$, otherwise $\mapsto 0$.

The homotopy connecting $\mathrm{in} \circ \rho_2$ to the identity $h_2 : \mathcal{C}\Big($  $\Big) \to \mathcal{C}\Big($  $\Big)$ such that $\delta_{s,t} \circ h_2 + h_2 \circ \delta_{s,t} = \mathrm{id} - \mathrm{in} \circ \rho_2$, is defined by the formulas:

(19) 

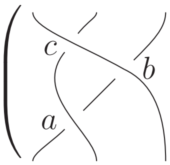
$\otimes [xab] \mapsto -$ 

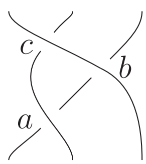
$\otimes [xb]$, 

$\otimes [xb] \mapsto$ 

$\otimes [x]$,

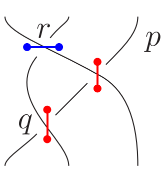
otherwise $\mapsto 0$.

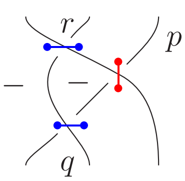
The homotopy connecting $\mathrm{in} \circ \rho_3$ to the identity, that is, a map $h_3 : \mathcal{C}\Big($ 

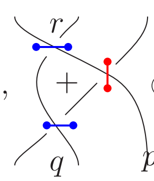
$\Big) \to \mathcal{C}\Big($ 

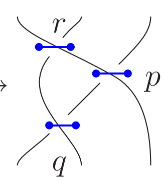
$\Big)$ such that $\delta_{s,t} \circ h_3 + h_3 \circ \delta_{s,t} = \mathrm{id} - \mathrm{in} \circ \rho_3$, is defined by the formulas:

(20)



$\otimes [xab] \mapsto -$ 

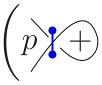
$\otimes [xb]$, 

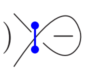
$\otimes [xb] \mapsto$ 

$\otimes [x]$,

otherwise $\mapsto 0$

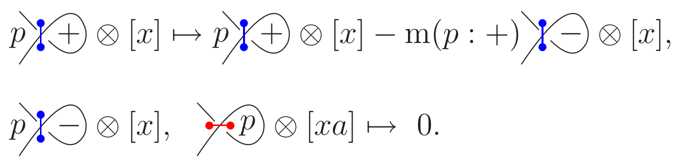
Appendix B. Retractions.

The retraction $\rho_{1'} : \mathcal{C}\Big($  $\Big) \to \mathcal{C}\Big(p$ 

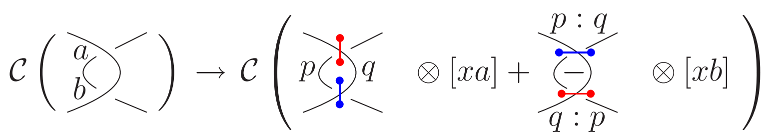
$\otimes [x] - \mathrm{m}(p:+)$ 

$\otimes [x]\Big)$ is defined by the formulas

(21)

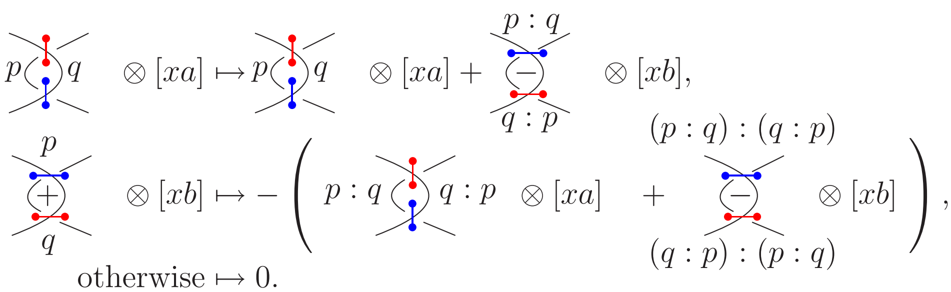


The retraction $\rho_2 : \mathcal{C}$

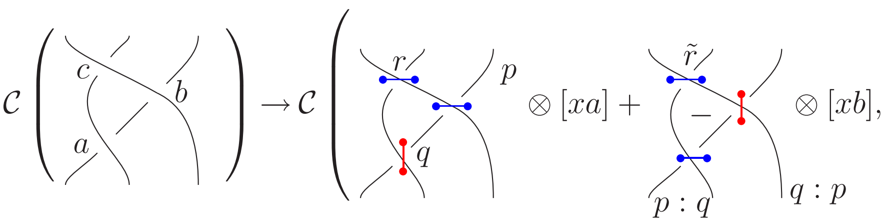


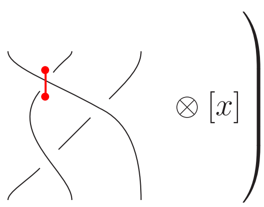
is defined by the formulas

(22)



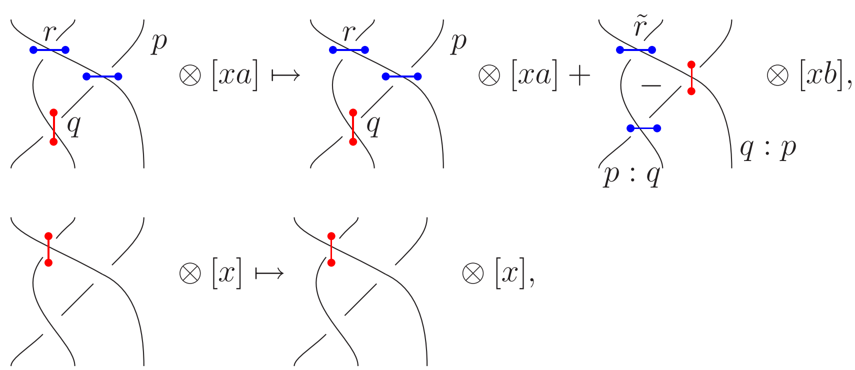
The retraction $\rho_3 : \mathcal{C}$

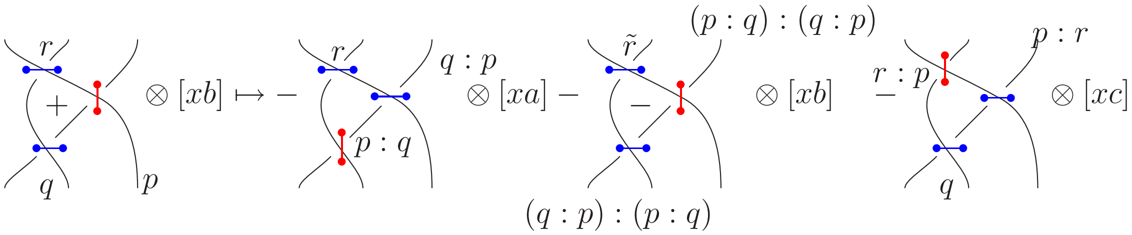


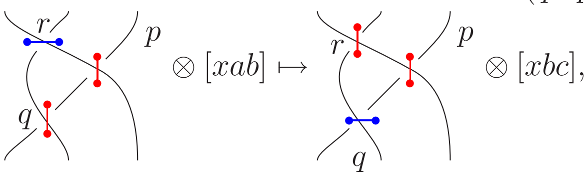


is defined by the formulas

(23)







otherwise $\mapsto 0$.

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[2] N. Ito, *Chain homotopy maps and a universal differential for Khovanov-type homology*, math.GT/0907.2104.

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Department of Pure and Applied Mathematics Waseda University. Tokyo 169-8555, Japan.
E-mail address: `noboru@moegi.waseda.jp`